UNIQUENESS OF THE DETERMINATION OF THE TEMPERATURE DEPENDENCE OF THERMOPHYSICAL PROPERTIES IN PIECEWISE MONOTONIC PROCESSES

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UDC 536.2:517.9

We investigate the uniqueness of the determination of the temperature dependence of the thermal conductivity and the volumetric heat capacity from known temperatures and heat fluxes on the boundaries.

In modeling high-intensity nonstationary thermal processes it is necessary to take account of the temperature dependence of the thermophysical properties of the material. However, this dependence cannot always be determined by classical methods. Thus, it is important to be able to determine these properties from directly measurable quantities by solving a certain inverse problem. Investigation of the uniqueness of the solution of such inverse problems involves the development of experimental arrangements which allow the determination of the temperature dependence of the thermophysical properties of the materials in high intensity nonstationary processes.

The problem of the uniqueness of the determination of the temperature dependence of the thermophysical properties when measurements are performed at certain points of a body has been treated in [1-7]. In those papers monotonic processes were considered, i.e., processes in which the heat flux does not change sign at the points of measurement or, what is more, the temperature is a monotonic function of time. However, the results obtained in [2-4] are easily carried over to piecewise monotonic processes, which are important in applications. We show this by the example of a problem treated in [2].

We use the notation $Q_\tau \equiv \{(x,\ t): 0<x<1,\ 0<t\leqslant\tau\}$, $Q\equiv Q_T$, $\bar{Q}_\tau \equiv \{(x,\ t): 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant\tau\}$ and consider the problem of determining the triple of functions {k(u), c(u), u(x, t)} which satisfy the equation

$$c(u)u_t=(k(u)u_x)_x,\quad (x,\ t)\in Q \tag{1}$$

and the conditions

$$\begin{gathered}u(x,\ 0)=u_0\equiv\mathrm{const},\ u_x(0,\ t)=0,\ k(u)u_x|_{x=1}=\psi(t),\ \psi(0)=0,\\ u(0,\ t)=f_1(t),\quad u(1,\ t)=f_2(t),\quad f_2'(t)>0,\quad 0<t<\eta_1,\quad \eta_2<t<T,\\ f_2'(t)<0,\quad \eta_1<t<\eta_2,\quad f_1(0)=f_2(0)=u_0,\ f_1'(0)=f_2'(0)=0\end{gathered}$$

for known $\psi(t),\ f_1(t),\ f_2(t)\in C^1[0,\ T],\ u_0$.

Definition: We call the triple of functions {k(u), c(u), u(x, t)} a solution of problem (1), (2) if

1) $u(x,\ t)\in C^{2,1}(\bar{Q})$;

2) $k(u)>0,\ c(u)>0,\ k(u)\in C^1[R_1,\ R_2]$, where $R_1=\min\limits_{\bar{Q}} u(x,\ t)$; $R_2=\max\limits_{\bar{Q}} u(x,\ t)$; $c(u)$ satisfies the Lipshits conditions;

3) {k(u), c(u), u(x, t)} satisfy relations (1) and (2).

Assuming that a solution of problem (1), (2) exists, we prove it is unique for k(u) and c(u) belonging to the class of piecewise-analytic functions.

Lemma: For a solution of problem (1), (2) to exist it is necessary that $f_1'(t)\geqslant 0$ and $\psi(t)\geqslant 0$ for $t\in[0,\ \eta_1]$, where $f_1'(t)>0$ and $\psi(t)>0$ on the set everywhere dense in $[0,\eta_1]$

Proof: We introduce the functions

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 1009-1013, December, 1985. Original article submitted May 17, 1985.

0022-0841/85/4906-1487 $09.50

$$a(u)=\int_{u_0}^{u} c(s)\,ds, \quad b(u)=\int_{u_0}^{u} k(s)\,ds.$$

Then, integrating the identity

$$0 \equiv \iint_{Q_{\eta_1}} \varphi_x\,[a(u)_t - b(u)_{xx}]\,dxdt,$$

by parts, we obtain the relation

$$\iint_{\bar{Q}_{\eta_1}} u_x\,[c(u)\,\varphi_t + k(u)\,\varphi_{xx}]\,dxdt = \int_0^1 a(u)_x\varphi\,\Big|_{t=0}^{t=\eta_1} dx + \int_0^{\eta_1} [b(u)_x\varphi_x - a(u)_t\varphi]\,\Big|_{x=0}^{x=1} dt, \tag{3}$$

which is valid for sufficiently smooth functions $\varphi(x, t)$. Passing to the limit (cf. e.g. [2]) we find that (3) is valid also for all $\varphi(x, t) \in C^{2,1}(\bar{Q}_{\eta_1})$. In (3) we take as $\varphi(x, t)$ the solution of the boundary-value problem

$$c(u)\,\varphi_t + k(u)\,\varphi_{xx} = 0 \text{ in } Q_{\eta_1}\quad \varphi(x,\ \eta_1) = 0, \quad 0 \leqslant x \leqslant 1, \tag{4}$$

$$\begin{gathered}\varphi(0,\ t) = 0, \quad \varphi_x(1,\ t) = \chi(t), \quad \chi(t) \in H^{1+\alpha}[0,\ \eta_1], \\ \chi(\eta_1) = \chi'(\eta_1) = 0, \quad \chi(t) \geqslant 0, \quad \chi(t) \not\equiv 0, \quad 0 \leqslant t \leqslant \eta_1,\end{gathered} \tag{5}$$

whose existence follows from [8] and the inequality $\varphi(1,\ t) \geqslant 0,\ \varphi(1,\ t) \not\equiv 0,\ 0 \leqslant t \leqslant \eta_1$. Then, since $f_2'(t) > 0$ for $t \in (0,\ \eta_1)$, we obtain from (3) the inequality

$$\int_0^{\eta_1} \psi(t)\,\chi(t)\,dt = \int_0^{\eta_1} c(f_2(t))\,f_2'(t)\,\varphi(1,\ t)\,dt > 0.$$

Since $\chi(t)$ is the derivative of a continuously differentiable function, and $\psi(t)$ is a continuous, $\psi(t) \geqslant 0$, and moreover there is no interval $(\alpha,\ \beta) \in [0,\ \eta_1]$ such that $\psi(t) \equiv 0$ for $t \in (\alpha,\ \beta)$. This means that any interval $(\alpha,\ \beta)$ contains a point $\bar{t}$ such that $\psi(\bar{t}) > 0$, and by virtue of continuity, $\psi(t) > 0$ also in a certain neighborhood of $\bar{t}$. Thus, the set of zeros of the function $\psi(t)$ is nowhere dense in $[0,\ \eta_1]$, and consequently $\psi(t) > 0$ on a set everywhere dense in $[0,\ \eta_1]$.

We now replace boundary conditions (5) by the conditions

$$\begin{gathered}\varphi(0,\ t) = \chi(t), \quad \varphi_x(1,\ t) = 0, \quad \chi(t) \in H^{1+\alpha}[0,\ \eta_1], \\ \chi(\eta_1) = \chi'(\eta_1) = 0, \quad \chi(t) \geqslant 0, \quad 0 \leqslant t \leqslant \eta_1\end{gathered} \tag{6}$$

and take for $\varphi(x, t)$ in (3) the solution of the problem (4), (6). The existence of a solution of this problem follows from [8] together with the inequality $\varphi(1, t) \geqslant 0, \varphi(1, t) \not\equiv 0$). Then we obtain the inequality

$$\int_0^{\eta_1} c(f_1(t))\,f_1'(t)\,\chi(t)\,dt = \int_0^{\eta_1} c(f_2(t))\,f_2'(t)\,\varphi(1,\ t)\,dt > 0.$$

From which it follows that $f_1'(t) \geqslant 0$ for $t \in [0,\ \eta_1]$ and $f_1'(t) > 0$ on a set everywhere dense in $[0,\ \eta_1]$.

Theorem. For k(u) and c(u) belonging to the class of piecewise analytic functions, the solution of problem (1), (2) is unique.

Proof. Using the lemma, we find that the conditions of Theorem 4 of [2] are satisfied $t \in [0,\ \eta_1]$ (the fact that in the lemma rigorous inequalities are valid on a set which is everywhere dense, while in Theorem 4 they hold over the whole interval, is not important) and thus the theorem was proved in [2] for $t \in [0,\ \eta_1]$, and consequently for $u \in [u_0,\ f_2(\eta_1)]$.

Let us now consider the interval $t \in (\eta_1, \eta_2)$, on which $f_2'(t) < 0$. Two cases are possible: either $f_2(\eta_2) \geqslant u_0$, or $f_2(\eta_2) < u_0$. In the first case the range of values of the function u(x, t) for $\eta_1 \leqslant t \leqslant \eta_2$ belongs to the segment $[u_0, f_2(\eta_2)]$, for which it is already known that k(u), c(u), and u(x, t) are uniquely determined. If $f_2(\eta_2) < u_0$, there exists an $\bar{\eta}$ such that $f_2(\bar{\eta}) = u_0$, $f_2(t) > u_0$ for $t \in (0, \bar{\eta})$ and $f_2(t) < u_0$ for $t \in (\bar{\eta}, \eta_2]$. We shall prove that then the minimum value of the function u(x, t) in the range $x \in [0, 1]$, $t \in [\bar{\eta}, \tau]$, $\tau \leqslant \eta_2$, occurs at the point (1, τ) where $f_1(\tau) > f_2(\tau)$. According to Lemma 1 of [1] $u_x(x, \eta_1) \geqslant 0$, and since $f_1'(t) \geqslant 0$, $f_1'(t) \not\equiv 0$ for $t \in [0, \eta_1]$, then $u(x, \eta_1) > u_0$. We extend Eq. (1) and its solution, which is an even function of x, to the segment [−1, 1]. Applying the rigorous maximum principle to the problem obtained (cf. Theorem 4 p. 57 and remark on p. 58 of [9]), we find that in the interval $x \in [-1, 1]$ $t \in [\eta_1, \tau]$ a maximum value of u(x, t) smaller than $f_1(\eta_1)$ cannot be reached for $|x| < 1$, and consequently it is reached for x = 1. Since $f_2(t)$ is monotonic for $t \in [\bar{\eta}, \tau]$, the minimum value of u(x, t) for $t \in [\bar{\eta}, \tau]$ is reached at the point (1, τ).

We now prove that $u_x(1, t) < 0$ for $t \in [\bar{\eta}, \eta_2]$. Since u(x, t) has a minimum at the point $(1, f_2(\eta_2))$, by theorem 14 on p. 69 of [9], $u_x(1, f_2(\eta_2)) < 0$. The statement of this theorem is valid for Eq. (1) without assuming that this minimum is negative. Since $\tau \in [\bar{\eta}, \eta_2)$ $u(1, \tau) \leqslant u(x, \tau)$, for $u_x(1, \tau) \leqslant 0$. We take ε > 0 such that $u_t(x, \tau) < 0$ for $x \in [1-\varepsilon, 1]$. Then for $x \in [1-\varepsilon, 1)$

$$k(u)\, u_x|_{(1,\,\tau)} - k(u)\, u_x|_{(x,\,\tau)} = \int_x^1 c(u(s,\ \tau))\, u_t(s,\ \tau)\, ds$$

and the assumption that $u_x(1, \tau) = 0$ leads to the inequality $u_x(x, \tau) > 0$ for $x \in [1-\varepsilon, 1)$. It follows from this inequality that $u(x, \tau) \geqslant u(1, \tau)$, and this contradicts the inequality $u(x, \tau) \geqslant u(1, \tau)$ proved earlier. Thus, $u_x(1, \tau) > 0$.

Thus, we have proved that for $\tau \in [\bar{\eta}, \eta_2]$, $u_x(1, \tau) > 0$ $u(1, \tau) = \min\limits_{\bar{Q}_\tau} u(x, t)$, and by hypothesis $f_2'(t) < 0$ for $t \in [\bar{\eta}, \eta_2)$ and $f_2'(\eta_2) = 0$. Consequently for $[\bar{\eta}, \eta_2)$ there exists a neighborhood G of the boundary x = 1 in which $u_x < 0$ and $u_t < 0$. Since $f_2(t) < f_1(t)$ for $t \in [\bar{\eta}, \eta_2]$, by repeating the proof of Theorem 4[2] we can choose a τ such that the set of points (x, t) for which $\Delta c(u) \neq 0$ and $\Delta k(u) \neq 0$ falls within the neighborhood G, and obtain the relation $I_\tau \equiv 0$ (cf. p. 400 [2]) for Δc(u) and Δk(u) having fixed signs. From now on the proof repeats that of Theorem 4 of [2]. For $t \in [\eta_1, \eta_2]$ the theorem is proved. For $t \in [\eta_2, T]$ the proof is similar.

It is clear that the theorem remains valid when $f_2'(t)$ does not change sign twice, and the derivative a finite number of times. We note also that the theorem is valid when the initial condition is not constant. In this case, however, it is necessary to require that the coefficients sought were known on a set of values giving the initial value of the function $u_0(x)$.

NOTATION

u, temperature; c(u), volumetric heat capacity; k(u), thermal conductivity; x, coordinate; t, time; T, duration of process; ′, symbol for derivative of a function of a single variable; u_0, initial temperature; $f_1(t)$ and $f_2(t)$, boundary values of temperature; τ, η_1, η_2, $\bar{\eta}$, $\bar{t}$, certain times; ψ(t), heat flux at boundary; $C^1[0, T]$, set of functions continuously differentiable on the segment [0, T]; $C^{1,2}[\bar{Q}]$, set of functions twice continuously differentiable with respect to the first variable, and once continuously differentiable with respect to the second variable on the set $\bar{Q}$; $H^{1+\alpha}[0, \eta_1]$, Holder class (cf. p. 16 of [4]); φ(x, t), solution of subsidiary problem; max u(x, t), maximum and minimum of function u(x, t) on the set $\bar{Q}$; α, β, ε, certain numbers; $\bar{Q}c(u)$ and Δk(u), differences of two volumetric heat capacities and thermal conductivities respectively (cf. [1]); I_τ, a certain integral (cf. p.400 [1]). Subscripts: x, t, symbols for derivative with respect to corresponding variable.

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TWO-DIMENSIONAL CONVERSE PROBLEMS FOR QUASILINEAR THERMAL CONDUCTIVITY EQUATIONS

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Converse problems on determination of unknown functions which depend on solution of the original problem and the spatial variable are studied.

One-dimensional converse problems involving unknown functions dependent on the solution of the original problem were considered in [1]. In view of the fact that solutions of converse problems are sought within special classes of functions, we will first define those classes.

Definition 1. We will say that a function q(u, x) belongs to the class $\mathfrak{M}_\beta[R_1, R_2]$, if $q(u, x) \in C^{3,2}([R_1, R_2] \times [0, \infty)) \cap C((-\infty, \infty) \times [0, \infty))$ and the following conditions are satisfied: $q_x'(u, x) \leqslant 0$ for $u \geqslant 0$, and for any two functions of the given class their difference $\tilde{q}(u, x) = q_1(u, x) - q_2(u, x)$ satisfies the inequality

$$\|\tilde{q}_x'(u, x)\|_u \leqslant c\|\tilde{q}(u, x)\|_u, \quad \left\|\frac{\partial^{k+m} q(u, x)}{\partial x^k \partial u^m}\right\| \leqslant \beta, \quad k+m \leqslant 2$$

, where c is a fixed constant.

Definition 2. The function $\sigma(u, x) \in \mathfrak{M}_\beta^k[R_1, R_2]$, if $\sigma(u, x) \in C^{3+k, 2+k}([R_1, R_2] \times [0, \infty)) \cap C^{k,k}((-\infty, \infty) \times [0, \infty))$, and the conditions $\left\|\frac{\partial^{k+m}\sigma(u, x)}{\partial x^k \partial u^m}\right\| \leqslant \beta$, $k+m \leqslant 2$, $0 < \nu \leqslant \sigma(u, x) \leqslant \mu$, are valid, and for any two functions of the given class their difference $\tilde{\sigma}(u, x) = \sigma_1(u, x) - \sigma_2(u, x)$ satisfies the inequality $\|\tilde{\sigma}_x'(u, x)\|_u \leqslant c\|\tilde{\sigma}(u, x)\|_u$.

We will now note some facts necessary for the future.

Lemma 1. Let $\varphi(t)$ be a continuous function at $0 \leqslant t \leqslant T$ and

$$\varphi(t) \leqslant \psi(t) + c\int_0^t \varphi(\tau)\left(1 + \frac{1}{\sqrt{t-\tau}}\right)d\tau, \quad 0 \leqslant t \leqslant T$$

, then for any t^*, $0 \leqslant t^* \leqslant T$,

$$\max_{0 \leqslant t \leqslant t^*} |\varphi(t)| \leqslant c_1 \max_{0 \leqslant t \leqslant t^*} |\psi(t)|,$$

where the constant c_1 depends on c and T.

Kuzbass Polytechnic Institute, Kemerovo. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 1013-1015, December, 1985. Original article submitted May 17, 1985.

 0022-0841/85/4906-1490$09.50